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A note on self-dual modules and Dedekind rings[1]

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Abstract

It is shown that a commutative noetherian ring is a finite direct sum of Dedekind rings and artinian uniserial rings if and only if every module of finite length is selfdual. A module of finite length is said to be selfdual if it is isomorphic to its dual with respect to the minimal injective cogenerator. © 1998 Elsevier Science B.V. All rights reserved.

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It is well-known that every finite abelian group is isomorphic to its character group. In this short note we show that this property characterizes Dedekind rings among commutative noetherian domains. For a commutative ring R we denote by E the minimal injective cogenerator of R, i.e., the injective hull of the direct sum of all isomorphism types of simple R-modules. As in the case of finite abelian groups, the character module M^* of an R-module M is defined as the group $Hom_R(M,E)$ which is also an R-module in the trivial way. Since every module M of finite length can be considered as a module over the factor ring $\bar{R}=R/\{r\in R \mid rM=0\}$ which is obviously a finite direct sum of local artinian rings, in view of Matlis duality M is reflexive, i.e., it is isomorphic to its second character module. Inspired by the above remark on character groups of abelian groups and the duality of modules of finite length we introduce the following notion.

Definition. A module M is said to be *self-dual* if it is isomorphic to its character module.

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Recall that a module is said to be *uniserial* if its submodules are totally ordered by inclusion. With the help of this notion we are now in position to state and to prove the main result.

Theorem. *A commutative noetherian ring R is a finite direct sum of uniserial artinian rings and Dedekind rings if and only if every module of finite length is self-dual.*

Proof. Assume that R is a finite direct sum of uniserial artinian rings and Dedekind rings. If an R-module M is of finite length, then by [2, Exercise 2, p. 216] or in view of [2, Theorem 2.5.1, p. 244] the factor ring $\bar{R}=R/I$ is a finite direct sum of uniserial artinian rings where I is the annihilator ideal of M. Therefore, M is a finite direct sum of uniserial modules of finite length. Consequently, we can assume that $\bar{R}$ is a uniserial artinian ring and M is also a uniserial $\bar{R}$-module. Hence $M^* = Hom_{\bar{R}}(M,\bar{R})$. Since both M and M^* are essential extensions of the simple $\bar{R}$-module, they are embedded into $\bar{R}$ and hence they are isomorphic by the equality of the lengths of M and M^*.

Conversely, assume that every module of finite length is self-dual. Let $\mathcal{M}$ be any maximal ideal of R and U be the injective hull of the simple R-module $R/\mathcal{M}$. To show that U is uniserial, it is enough to prove that the factor ring $A=R/M^2$ is uniserial. Let V be the annihilator of M^2 in U. Since V is the dual of A, V is isomorphic to A which implies immediately that M/M^2 is simple. Thus, A is a uniserial ring and hence U is also uniserial. If U is of finite length, then $U=Rx$ for some $x\in U$. Since U is also the minimal injective cogenerator of the localization $R_{\mathcal{M}}$ by Theorem 18.4 in [4], we obtain $U=Rx=R_{\mathcal{M}}x$, i.e., $R_{\mathcal{M}}\cong U$. Since the kernel I of the canonical ring homomorphism from R into $R_{\mathcal{M}}$ is the ideal $\{r\in R \mid \exists s\notin \mathcal{M} : rs=0\}$ and R is a noetherian ring, there are elements $r_1,\dots,r_n\in R$ such that $I=\sum Rr_i$. For each r_i there is $a_i\in R\backslash\mathcal{M}$ with $r_i a_i=0$. Put $a=\prod a_i$. Since $\mathcal{M}$ is a prime ideal, $a\notin\mathcal{M}$ and $r_i a=0$ for all indices i. Hence

$$I=\sum Rr_i\subset ann_R\, a=\{r\in R \mid ra=0\}\subset\{r\in R \mid \exists s\notin\mathcal{M} \mid rs=0\}=I,$$

i.e., $I=ann_R\, a$. Therefore $U=Rx=R_{\mathcal{M}}x\cong Ra$. Consequently, $Ra\subset R$ is an injective R-module. This shows that R is a direct sum of the ring Ra and the ring S where Ra is a uniserial artinian ring and S satisfies also the condition that every S-module of finite length is self-dual. Since R is a noetherian ring, the above result shows that there are only finitely many maximal ideals $\mathcal{M}$ such that the injective hull of the simple module $R/\mathcal{M}$ is a uniserial module of finite length. Therefore, R is a finite direct sum of uniserial artinian rings and the ring T such that the injective hulls of simple T-modules are uniserial artinian modules of infinite length. This implies that for every maximal ideal $\mathcal{M}$ of T the localization $T_{\mathcal{M}}$ is a domain, in fact it is DVR. Consequently, by Theorem 168 in [3], T is a finite direct sum of domains. Hence, T is a finite direct sum of Dedekind domains which completes the proof. □

Corollary. *A noetherian domain is a Dedekind ring if and only if modules of finite length are self-dual.*

Remark. It might be interesting to describe all rings satisfying the conditon that modules of finite length are self-dual. This class of rings contains, for example, all von Neumann regular rings or more generally all locally noetherian rings, i.e., rings such that their localizations at maximal ideals are noetherian. Moreover, all rings such that their maximal ideals are idempotent, belong to this class, too.

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